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Generalized Fundamental Equations on the Submanifolds of a Manifold ESX_n

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A connection which is both Einstein and semisymmetric is called an *ES* connection, and a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by a unified field tensor $g_{\lambda\mu}$ through an *ES* connection is called an n-dimensional *ES* manifold and denoted by ESX_n. We investigate some necessary and sufficient conditions for submanifolds of ESX_n to be also Einstein and derive the generalized fundamental equations on various submanifolds of ESX_n, such as generalized Gauss formulas, generalized Weingarten equations, and generalized Gauss–Codazzi equations. We employ the useful and powerful concept of C-nonholonomic frame of reference, introduced in earlier work.

1. INTRODUCTION

Einstein (1950, Appendix II) proposed a unified field theory which, while physically motivated, consists mainly of a set of geometrical postulates for the space-time X_4, the consequences of which he did not pursue extensively.

Characterizing Einstein's unified field theory as a set of geometrical postulates in X_4, Hlavatý (1957) provided its mathematical foundation. Since then the geometrical consequences of these postulates have been developed by a number of mathematicians.

Generalizing X_4 to the n-dimensional generalized Riemannian manifold X_n, the n-dimensional generalization of Einstein's unified field theory has been studied by Wrede (1958), Mishra (1959), Chung and Han (1981), and Chung and Cheoi (1985). The latter two references particularly investigated the n-dimensional generalization of Principle A using recurrence relations.

Recently, Chung and Cho (1987) introduced the concept of the n-dimensional *ES* manifold (denoted by ESX_n), imposing the semisymmetric

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condition (2.9) given below, on X_n, and found the unique representation of Einstein's connection in a beautiful and surveyable form, (2.10). Many results concerning the manifold ESX_n have been obtained, such as conformal change between ES manifolds (Chung and Cho, 1987), curvature tensors and unified field equations on ESX_n (Chung and Lee, 1988), and generalized fundamental equations for the hypersubmanifolds of ESX_n (Chung and Lee, 1989). In particular, the new concept of C-nonholonomic frame of reference introduced by Chung *et al.* (1989) is a very powerful tool in the study of the geometry of submanifolds of ESX_n.

The purpose of the present paper is to derive generalized fundamental equations on the submanifolds of ESX_n, employing the C-nonholonomic frame of reference. Briefly, the organization of the present paper is as follows. Section 2 introduces some preliminary concepts, results, and notations. Section 3 is devoted to the derivation of several useful identities which hold on the submanifolds of X_n and ESX_n. In particular, we investigate some necessary and sufficient conditions for the submanifolds to be also Einstein. In Section 4 we derive the generalized fundamental equations on the submanifolds of an ESX_n—generalized Gauss formulas, generalized Weingarten equations, and generalized Gauss–Codazzi equations. They will be presented in surveyable and refined forms. In Section 5 the previous results are specialized to two special submanifolds of ESX_n, hypersubmanifolds and tangential submanifolds defined to be those to which the ES vector is tangential. We note that the fundamental equations of hypersubmanifolds of ESX_n coincide with those obtained by Chung and Lee (1989).

All considerations in the present paper deal with the general case $n \geq 2$ and all possible classes and indices of inertia.

2. PRELIMINARIES

This section is a brief collection of definitions, notations, and basic results used in subsequent considerations. The detailed proofs are given in Chung and Cho (1987), Chung and Lee (1989), Chung *et al.* (1989), and Hlavatý (1957).

2.1. The Manifolds X_n

The usual Einstein n-dimensional unified field theory is based on a generalized n-dimensional Riemannian manifold X_n, a generalization of the space-time X_4, which is referred to a real coordinate system y^ν and obeys coordinate transformations[2] $y^\nu \to \bar{y}^\nu$ for which $\mathrm{Det}(\partial \bar{y}/\partial y) \neq 0$.

[2]Throughout the paper, Greek indices are used for the holonomic components of tensors in X_n. They take the values $1', \ldots, n'$ and follow the summation convention.

The algebraic structure on X_n is imposed by a general nonsymmetric tensor $g_{\lambda\mu}$, called the *unified field tensor*. It may be split into a symmetric part $h_{\lambda\mu}$ and a skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu}=h_{\lambda\mu}+k_{\lambda\mu} \tag{2.1}$$

where

$$\mathfrak{g}=\mathrm{Det}(g_{\lambda\mu})\neq 0, \qquad \mathfrak{h}=\mathrm{Det}(h_{\lambda\mu})\neq 0 \tag{2.2}$$

We may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu}=\delta_\mu^\nu \tag{2.3}$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of holonomic components of tensors in X_n in the usual manner.

The differential geometric structure on X_n is imposed by the tensor $g_{\lambda\mu}$ by means of a real connection $\Gamma^\nu_{\lambda\mu}$, which satisfies the transformation rule

$$\bar{\Gamma}^\nu_{\lambda\mu}=\frac{\partial\bar{y}^\nu}{\partial y^\alpha}\left(\frac{\partial y^\beta}{\partial\bar{y}^\lambda}\frac{\partial y^\gamma}{\partial\bar{y}^\mu}\Gamma^\alpha_{\beta\gamma}+\frac{\partial^2 y^\alpha}{\partial\bar{y}^\lambda\,\partial\bar{y}^\mu}\right) \tag{2.4}$$

and the system of Einstein equations

$$\partial_\omega g_{\lambda\mu}-\Gamma^\alpha_{\lambda\omega}g_{\alpha\mu}-\Gamma^\alpha_{\omega\mu}g_{\lambda\alpha}=0 \tag{2.5a}$$

or equivalently

$$D_\omega g_{\lambda\mu}=2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} \tag{2.5b}$$

Here D_ω denotes the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^\nu=\Gamma^\nu_{[\lambda\mu]} \tag{2.6}$$

is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$.

The connection $\Gamma^\nu_{\lambda\mu}$ will be called *Einstein*, since it is a solution of (2.5). Thus, our manifold X_n is endowed with a unified tensor field $g_{\lambda\mu}$ in the first and is connected by an Einstein connection $\Gamma^\nu_{\lambda\mu}$ in the second.

A procedure similar to Christoffel's elimination applied to the symmetric part of (2.5b) yields that if the system (2.5) admits a solution $\Gamma^\nu_{\lambda\mu}$, it must be of the form (Hlavatý, 1957)

$$\Gamma^\nu_{\lambda\mu}=\left\{{}^{\,\nu}_{\lambda\mu}\right\}+S_{\lambda\mu}{}^\nu+U^\nu{}_{\lambda\mu} \tag{2.7}$$

where $\left\{{}^{\,\nu}_{\lambda\mu}\right\}$ are the Christoffel symbols with respect to $h_{\lambda\mu}$ and

$$U^\nu{}_{\lambda\mu}=2h^{\nu\alpha}S_{\alpha(\lambda}{}^\beta k_{\mu)\beta}=2k_{\beta(\lambda}S_{\mu)}{}^{\nu\beta} \tag{2.8}$$

2.2. The Manifolds ESX_n

A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be *semisymmetric* if its torsion tensor is of the form

$$S_{\lambda\mu}{}^{\nu} = 2\delta^{\nu}_{[\lambda}X_{\mu]} \tag{2.9}$$

for an arbitrary vector $X_\mu \neq 0$. A connection which is both semisymmetric and Einstein is called an *ES connection*. An *n-dimensional ES manifold*, denoted by ESX_n, is a manifold X_n on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through the *ES* connection $\Gamma^{\nu}_{\lambda\mu}$.

It has been shown that the *ES* connection $\Gamma^{\nu}_{\lambda\mu}$ must be of the form (Chung and Cho, 1987)

$$\Gamma^{\nu}_{\lambda\mu} = \left\{{}^{\,\nu}_{\lambda\mu}\right\} + 2k_{(\lambda}{}^{\nu}X_{\mu)} + 2\delta^{\nu}_{[\lambda}X_{\mu]} \tag{2.10}$$

It has also been shown that in an X_n there always exists a uniquely determined *ES* connection $\Gamma^{\nu}_{\lambda\mu}$ with a unique *ES* vector X_μ satisfying (Chung and Cho, 1987)

$$\nabla_\omega k_{\lambda\mu} + P_{\omega[\lambda}X_{\mu]} = 0 \tag{2.11}$$

Here

$$P_{\lambda\mu} = {}^{(2)}k_{\lambda\mu} - h_{\lambda\mu}, \qquad {}^{(2)}k_{\lambda\mu} = k_\lambda{}^\alpha k_{\alpha\mu} \tag{2.12a}$$

is a symmetric tensor with

$$\operatorname{Det}(P_{\lambda\mu}) \neq 0 \tag{2.12b}$$

2.3. The *C*-Nonholonomic Frame of Reference in X_n at Points of X_m

This section deals with a brief introduction of the concept of the *C*-nonholonomic frame of reference in X_n at points of its submanifold X_m, $m<n$ (Chung *et al.*, 1989).

Agreement 2.1. In our further considerations in the present paper, we use the following types of indices:

(a) Lowercase Greek indices α, β, γ, . . . , running from 1 to n and used for the holonomic components of tensors in X_n.

(b) Capital Latin indices A, B, C, . . . , running from 1 to n and used for the *C*-nonholonomic components of tensors in X_n at points of X_m.

(c) Lowercase Latin indices i, j, k, . . . , with the exception of x, y, and z, running from 1 to m ($<n$).

(d) Lowercase Latin italic indices x, y, and z, running from $m+1$ to n.

The summation convention is operative with respect to each set of the above indices within their range, with the exception of x, y, and z.

Let X_m be a submanifold of X_n defined by a system of sufficiently differentiable equations

$$y^\nu = y^\nu(x^1, \ldots, x^m) \tag{2.13}$$

where the matrix of derivatives $B_i^\nu = \partial y^\nu / \partial x^i$ is of rank m.

At each point of X_m there exists the *first set* $\{B_i^\nu, \underset{x}{N}^\nu\}$ of n linearly independent nonnull vectors. The m vectors B_i^ν are tangential to X_m and the $n-m$ vectors $\underset{x}{N}^\nu$ are normal to X_m and mutually orthogonal. That is,

$$h_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta = 0, \qquad h_{\alpha\beta} \underset{x}{N}^\alpha \underset{y}{N}^\beta = 0 \qquad \text{for} \quad x \neq y \tag{2.14a}$$

The process of determining the set $\{\underset{x}{N}^\nu\}$ is not unique unless $m = n-1$. However, we may choose their magnitudes such that

$$h_{\alpha\beta} \underset{x}{N}^\alpha \underset{x}{N}^\beta = \varepsilon_x \tag{2.14b}$$

where $\varepsilon_x = \pm 1$ according as the left-hand side of (2.14b) is positive or negative.

Put

$$E_A^\nu = \begin{cases} B_i^\nu, & \text{if} \quad A = 1, \ldots, m \quad (=i) \\ \underset{x}{N}^\nu, & \text{if} \quad A = m+1, \ldots, n \quad (=x) \end{cases} \tag{2.15}$$

Corresponding to the first set $\{E_A^\nu\}$ of n linearly independent vectors, there exists a unique *second set* $\{E_\lambda^A\}$ of linearly independent vectors at points of X_m such that

$$E_\lambda^A E_A^\nu = \delta_\lambda^\nu, \qquad E_\alpha^A E_B^\alpha = \delta_B^A \tag{2.16}$$

Putting

$$E_\lambda^A = \begin{cases} B_\lambda^i, & \text{if} \quad A = 1, \ldots, m \quad (=i) \\ \overset{x}{N}_\lambda, & \text{if} \quad A = m+1, \ldots, n \quad (=x) \end{cases} \tag{2.17}$$

We note that the vectors B_λ^i and $\overset{x}{N}_\lambda$ are also tangential and normal, respectively, to X_m in virtue of Theorem 2.3.

Now, we are ready to introduce the following concepts of C-nonholonomic frame of reference and induced tensors.

Definition 2.2. The sets $\{E_A^\nu\}$ and $\{E_\lambda^A\}$ will be referred to as the *C-nonholonomic frame of reference* in X_n at points of X_m. This frame gives rise to *C-nonholonomic components* of tensors in X_n. If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor in X_n, then at points of X_m its C-nonholonomic components $T^{A\cdots}_{B\cdots}$ are defined by

$$T^{A\cdots}_{B\cdots} = T^{\alpha\cdots}_{\beta\cdots} E^{A}_{\alpha} \cdots E^{\beta}_{B} \cdots \tag{2.18a}$$

In particular, the quantities

$$T^{i\cdots}_{j\cdots} = T^{\alpha\cdots}_{\beta\cdots} B^{i}_{\alpha} \cdots B^{\beta}_{j} \cdots \tag{2.19}$$

are components of a tensor in X_m and are called the components of the *induced tensor* of $T^{\nu\cdots}_{\lambda\cdots}$ on X_m of X_n.

In virtue of (2.16), an easy inspection shows that

$$T^{\nu\cdots}_{\lambda\cdots} = T^{A\cdots}_{B\cdots} E^{\nu}_{A} \cdots E^{B}_{\lambda} \cdots \tag{2.18b}$$

The following theorems and remark are consequences of the powerful C-nonholonomic frame of reference.

Theorem 2.3. The tensors B^{ν}_{i}, B^{i}_{λ}, $\underset{x}{N}^{\nu}$, $\overset{x}{N}_{\lambda}$, and

$$B^{\nu}_{\lambda} = B^{i}_{\lambda} B^{\nu}_{i} \tag{2.20}$$

are involved in the following identities:

$$B^{i}_{\alpha} B^{\alpha}_{j} = \delta^{i}_{j}, \qquad \overset{x}{N}_{\alpha} \underset{y}{N}^{\alpha} = \delta^{x}_{y}, \qquad B^{i}_{\alpha} \underset{x}{N}^{\alpha} = \overset{x}{N}_{\alpha} B^{\alpha}_{i} = 0 \tag{2.21}$$

$$B^{i}_{\lambda} = B^{\alpha}_{j} h_{\lambda\alpha} h^{ij}, \qquad \overset{x}{N}_{\lambda} = \varepsilon_x \underset{x}{N}_{\lambda} \tag{2.22a}$$

$$B^{\nu}_{i} = B^{j}_{\alpha} h^{\nu\alpha} h_{ij}, \qquad \underset{x}{N}^{\nu} = \varepsilon_x \overset{x}{N}^{\nu} \tag{2.22b}$$

$$h^{\nu\alpha} B^{i}_{\alpha} = h^{ij} B^{\nu}_{j}, \qquad h_{\lambda\alpha} B^{\alpha}_{i} = h_{ij} B^{j}_{\lambda} \tag{2.23}$$

$$B^{\nu}_{\lambda} = \delta^{\nu}_{\lambda} - \sum_{x} \overset{x}{N}_{\lambda} \underset{x}{N}^{\nu} \tag{2.24a}$$

$$B^{\alpha}_{\lambda} \overset{x}{N}_{\alpha} = B^{\nu}_{\alpha} \underset{x}{N}^{\alpha} = 0 \tag{2.24b}$$

$$B^{\alpha}_{\lambda} B^{i}_{\alpha} = B^{i}_{\lambda}, \qquad B^{\nu}_{\alpha} B^{\alpha}_{i} = B^{\nu}_{i}, \qquad B^{\nu}_{\alpha} B^{\alpha}_{\lambda} = B^{\nu}_{\lambda} \tag{2.24c}$$

Theorem 2.4. At each point of X_m any vector X_λ in X_n may be expressed as the sum of two vectors $X_i B^{i}_{\lambda}$ and $\sum_x X_x N_\lambda$, the former tangential to X_m

and the latter normal to X_m. That is,

$$X_\lambda = X_i B_\lambda^i + \sum_x X_x \overset{x}{N}_\lambda \tag{2.25a}$$

or equivalently

$$X^\nu = X^i B_i^\nu + \sum_x X^x \underset{x}{N}^\nu \tag{2.25b}$$

where

$$X_i = X_\alpha B_i^\alpha, \qquad X_x = X_\alpha \underset{x}{N}^\alpha, \qquad X_x = \varepsilon_x X^x$$
$$X^i = X^\alpha B_\alpha^i, \qquad X^x = X^\alpha \overset{x}{N}_\alpha \tag{2.26}$$

Furthermore, X_i (X^i) are components of a tangent vector relative to the transformations of X_m, while X_x (X^x) is invariant relative to the transformations of X_m and X_n.

Theorem 2.5. The C-nonholonomic components h_{AB} of $h_{\lambda\mu}$ and h^{AB} of $h^{\lambda\nu}$ are given by the matrix equations

$$(h_{AB}) = \begin{pmatrix} h_{11} & \cdots & h_{1m} & & & \\ \vdots & & \vdots & & 0 & \\ h_{m1} & \cdots & h_{mm} & & & \\ & & & \varepsilon_{m+1} & & \\ & 0 & & & \ddots & \\ & & & & & \varepsilon_n \end{pmatrix} \tag{2.27a}$$

$$(h^{AB}) = \begin{pmatrix} h^{11} & \cdots & h^{1m} & & & \\ \vdots & & \vdots & & 0 & \\ h^{m1} & \cdots & h^{mm} & & & \\ & & & \varepsilon_{m+1} & & \\ & 0 & & & \ddots & \\ & & & & & \varepsilon_n \end{pmatrix} \tag{2.27b}$$

Remark 2.6. The induced tensor g_{ij} of $g_{\lambda\mu}$ is given by

$$g_{ij} = g_{\alpha\beta} B_i^\alpha B_j^\beta \tag{2.28a}$$

where its symmetric part h_{ij} and skew-symmetric part k_{ij} are

$$h_{ij} = h_{\alpha\beta} B_i^\alpha B_j^\beta, \qquad k_{ij} = k_{\alpha\beta} B_i^\alpha B_j^\beta \tag{2.28b}$$

so that

$$g_{ij}=h_{ij}+k_{ij} \tag{2.29}$$

In the present paper, we restrict ourselves to submanifolds for which the following condition holds:

$$\mathrm{Det}(h_{ij})\neq 0 \tag{2.30}$$

In virtue of the condition (2.30), we may define a unique inverse tensor $\bar{h}^{ik}$ of h_{ij} by

$$h_{ij}\bar{h}^{ik}=\delta_j^k \tag{2.31}$$

It has been shown that $\bar{h}^{ik}$ is the induced tensor h^{ik} of $h^{\lambda\nu}$. That is, $\bar{h}^{ik}=h^{ik}$. Therefore, *the tensors* h_{ij} *and* h^{ij} *may be used for raising and/or lowering indices of the induced tensors on* X_m *in the usual manner.*

2.4. The Induced Connection on X_m of X_n

Definition 2.7. If $\Gamma_{\lambda\mu}^{\nu}$ is a connection on X_n, the connection Γ_{ij}^k defined by

$$\Gamma_{ij}^k=B_\gamma^k(B_{ij}^\gamma+\Gamma_{\alpha\beta}^\gamma B_i^\alpha B_j^\beta), \qquad B_{ij}^\gamma=\frac{\partial B_i^\gamma}{\partial x^j}=\frac{\partial^2 y^\gamma}{\partial x^i\,\partial x^j} \tag{2.32}$$

is called the *induced connection* of $\Gamma_{\lambda\mu}^{\nu}$ on X_m of X_n.

It should be remarked that the torsion tensor $S_{ij}{}^k$ of the induced connection Γ_{ij}^k is the induced tensor of the torsion tensor $S_{\lambda\mu}{}^\nu$ of the connection $\Gamma_{\lambda\mu}^\nu$. That is,

$$S_{ij}{}^k=S_{\alpha\beta}{}^\gamma B_i^\alpha B_j^\beta B_\gamma^k \tag{2.33}$$

Furthermore, the induced connection $\{{}^{\,k}_{ij}\}$ of $\{{}^{\,\nu}_{\lambda\mu}\}$ is the Christoffel symbol defined by h_{ij}. That is,

$$\{{}^{\,k}_{ij}\}=\tfrac{1}{2}h^{kp}(\partial_i h_{jp}+\partial_j h_{ip}-\partial_p h_{ij}) \tag{2.34}$$

3. ANALYSIS ON THE SUBMANIFOLDS X_n AND ESX_n

The section is devoted to the derivation of several identities which hold on the submanifolds of X_n and ESX_n. In particular, we prove the generalized Gauss formulas and find some necessary and sufficient conditions for the submanifolds of X_n to be Einstein.

In our subsequent considerations, we frequently use the following C-nonholonomic components:

$$k_{ix} = -k_{xi} = k_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta = g_{\alpha\beta} B_i^\alpha \underset{x}{N}^\beta \tag{3.1a}$$

$$S_{ij}{}^x = -S_{ji}{}^x = S_{\alpha\beta}{}^\gamma B_i^\alpha B_j^\beta \overset{x}{N}_\gamma \tag{3.1b}$$

$$U^x{}_{ij} = U^x{}_{ji} = U^\gamma{}_{\alpha\beta} B_i^\alpha B_j^\beta \overset{x}{N}_\gamma \tag{3.1c}$$

3.1. The Tensors $\overset{x}{\Omega}_{ij}$ and the Generalized Gauss Formulas

Let $\overset{0}{D}_j$ be the symbolic vector of the generalized covariant derivative with respect to x's. That is,

$$\overset{0}{D}_j B_i^\alpha = B_{ij}^\alpha + \Gamma_{\beta\gamma}^\alpha B_i^\alpha B_j^\beta - \Gamma_{ij}^k B_k^\alpha \tag{3.2}$$

Theorem 3.1. The vector $\overset{0}{D}_j B_i^\alpha$ in X_n is normal to X_m and is given by

$$\overset{0}{D}_j B_i^\alpha = -\sum_x \overset{x}{\Omega}_{ij} \underset{x}{N}^\alpha \tag{3.3}$$

where

$$\overset{x}{\Omega}_{ij} = -(\overset{0}{D}_j B_i^\alpha) \overset{x}{N}_\alpha \tag{3.4}$$

Proof. In virtue of (2.32), multiplication by B_α^m on both sides of (3.2) shows that $\overset{0}{D}_j B_i^\alpha$ is normal to X_m. The relation (3.4) follows from (3.3) by multiplying by $\overset{y}{\underset{x}{N}}_\alpha$ on both sides of (3.3) and making use of (2.21).

The tensor Ω_{ij} will be called the *generalized coefficients of the second fundamental form of* X_m.

Theorem 3.2. The tensors $\overset{x}{\Omega}_{ij}$ are the induced tensors of $D_\beta \overset{x}{N}_\alpha$ on X_m of X_n. That is,

$$\overset{x}{\Omega}_{ij} = (D_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta \tag{3.5}$$

Proof. Substituting (3.2) into (3.4) and making use of (2.21) and the relation

$$0 = \partial_j (B_i^\alpha \overset{x}{N}_\alpha) = B_{ij}^\alpha \overset{x}{N}_\alpha + (\partial_\beta \overset{x}{N}_\alpha) B_i^\alpha B_j^\beta$$

we have (3.5).

Let

$$\overset{x}{\Lambda}_{ij}=(\nabla_\beta \overset{x}{N}_\alpha)B_i^\alpha B_j^\beta \tag{3.6}$$

be the generalized coefficients of the second fundamental form with respect to the Christoffel symbols $\{{}_{\lambda\mu}^{\nu}\}$. Here ∇_β denotes the symbolic vector of the covariant derivative with respect to $\{{}_{\lambda\mu}^{\nu}\}$.

Theorem 3.3. The coefficients $\overset{x}{\Omega}_{ij}$ of the submanifold X_m are given by

$$\overset{x}{\Omega}_{ij}=\overset{x}{\Lambda}_{ij}-S_{ij}{}^x-U^x{}_{ij} \qquad \text{(on an } X_m \text{ of } X_n) \tag{3.7a}$$

$$\Omega_{ij}=\Lambda_{ij}-2\varepsilon_x X_{(i}k_{j)x} \qquad \text{(on an } X_m \text{ of } ESX_n) \tag{3.7b}$$

Proof. In virtue of (2.7), (3.5), (3.6), and (3.1), the relation (3.7a) follows:

$$\begin{aligned}\overset{x}{\Omega}_{ij}&=(D_\beta \overset{x}{N}_\alpha)B_i^\alpha B_j^\beta\\ &=[\partial_\beta \overset{x}{N}_\alpha-(\{{}_{\alpha\beta}^{\gamma}\}+S_{\alpha\beta}{}^\gamma+U^\gamma{}_{\alpha\beta})\overset{x}{N}_\gamma]B_i^\alpha B_j^\beta\\ &=\overset{x}{\Lambda}_{ij}-S_{ij}{}^x-U^x{}_{ij}\end{aligned}$$

In virtue of (2.9), (2.22), and (3.1), on an X_m of ESX_n we have

$$S_{ij}{}^x=0, \qquad U^x{}_{ij}=2X_{(i}k_{j)}{}^x=2\varepsilon_x X_{(i}k_{j)x} \tag{3.8}$$

since $k_j{}^x=\varepsilon_x k_{jx}$. Now, the substitution of (3.8) into (3.7a) gives (3.7b).

Now, we are ready to state the following generalized Gauss formulas for submanifolds of X_n and ESX_n. They are direct consequences of (3.3) and (3.7).

Theorem 3.4. The tensor $\overset{0}{D}_j B_i^\alpha$ satisfies the following identities:

$$\overset{0}{D}_j B_i^\alpha=-\sum_x(\overset{x}{\Lambda}_{ij}-S_{ij}{}^x-U^x{}_{ij})\underset{x}{N}^\alpha \tag{3.9}$$

(Generalized Gauss formulas for an X_m of X_n)

$$\overset{0}{D}_j B_i^\alpha=\sum_x(-\overset{x}{\Lambda}_{ij}+\varepsilon_x X_{(i}k_{j)x})\underset{x}{N}^\alpha \tag{3.10}$$

(Generalized Gauss formulas for an X_m of ESX_n)

Remark 3.5. Note that the tensor $\overset{x}{\Lambda}_{ij}$ defined by (3.6) is symmetric. Therefore, in virtue of (3.7), we see that $\overset{x}{\Omega}_{ij}$ is symmetric on an X_m of ESX_n, while it is not symmetric on an X_m of a general X_n.

3.2. Analysis on the Submanifolds of X_n

In this section, we prove several relations which hold on the submanifolds X_m of X_n and present two necessary and sufficient conditions for the induced connection to be an ES connection.

Theorem 3.6. On an X_m of X_n the induced connection Γ^k_{ij} of $\Gamma^\nu_{\lambda\mu}$ is of the form

$$\Gamma^k_{ij} = \left\{ {}^{\,k}_{ij} \right\} + S_{ij}{}^k + U^k{}_{ij} \tag{3.11}$$

where $S_{ij}{}^k$ and $U^k{}_{ij}$ are respectively the induced tensors of $S_{\lambda\mu}{}^\nu$ and $U^\nu{}_{\lambda\mu}$.

Proof. In virtue of (2.19) and (2.32), our assertion (3.11) may be obtained by substituting (2.7) into (2.32).

Although the tensor $U^\nu{}_{\lambda\mu}$ takes the form (2.8), its induced tensor $U^k{}_{ij}$ does not take the same form in general.

Theorem 3.7. On an X_m of X_n, the induced tensor $U^k{}_{ij}$ is of the form

$$U^k{}_{ij} = 2h^{kq} S_{q(i}{}^p k_{j)p} = 2k_{p(i} S_{j)}{}^{kp} \tag{3.12}$$

if and only if the following condition holds:

$$\sum_x S_{(j}{}^{kx} k_{i)x} = 0 \qquad \text{or} \qquad \sum_x k_{x(i} S_{j)k}{}^x = 0 \tag{3.13}$$

Proof. In virtue of (2.8), (2.19), (2.24a), and (3.1), we have

$$U^k{}_{ij} = U^\nu{}_{\lambda\mu} B^\lambda_i B^\mu_j B^k_\nu = 2k_{\beta(\lambda} S_{\mu)}{}^{\nu\beta} B^\lambda_i B^\mu_j B^k_\nu$$

$$2k_{p(i} S_{j)}{}^{kp} = 2k_{\beta(\lambda} S_{\mu)}{}^{\nu\varepsilon} B^\beta_\varepsilon B^\lambda_i B^\mu_j B^k_\nu$$

$$= U^k{}_{ij} + 2\sum_x S_{(j}{}^{kx} k_{i)x}$$

from which our assertion follows.

Remark 3.8. The following statements are direct consequences of Theorem (3.7).

(a) On a submanifold X_m of ESX_n, the tensor $U^k{}_{ij}$ always takes the form

$$U^k{}_{ij} = 2h^{kq} S_{q(i}{}^p k_{j)p} = 2k_{(i}{}^k X_{j)} \tag{3.14}$$

since the condition (3.13) holds in virtue of (3.8).

(b) A manifold X_n is called an *EM manifold* if it is connected by an Einstein connection $\Gamma^{\nu}_{\lambda\mu}$ of the form

$$\Gamma^{\nu}_{\lambda\mu}=\left\{{}^{\nu}_{\lambda\mu}\right\}+2\delta^{\nu}_{\lambda}X_{\mu}-2g_{\lambda\mu}X^{\nu} \tag{3.15}$$

for an arbitrary vector X_{λ}. In this case, the relations (2.18a), (3.15), and (2.21) give

$$\begin{aligned} S_{jk}{}^{x}=S_{\lambda\mu}{}^{\nu}B^{\lambda}_{j}B^{\mu}_{k}\overset{x}{N}_{\nu}&=2(\delta^{\nu}_{[\lambda}X_{\mu]}-k_{\lambda\mu}X^{\nu})B^{\lambda}_{j}B^{\mu}_{k}\overset{x}{N} \\ &=-2k_{jk}X^{x} \end{aligned}$$

Hence, a necessary and sufficient condition for the tensor $U^{k}{}_{ij}$ to be of the form (3.12) is

$$0=\sum_{x}k_{x(i}S_{j)k}{}^{x}=-2\sum_{x}X^{x}k_{x(i}k_{j)k}$$

One such case is that the vector X_{λ} be tangential to X_m (i.e., $X^{x}=0$).

Theorem 3.9. On an X_m of X_n, the induced tensor of $D_{\omega}g_{\lambda\mu}$ may be given by

$$(D_{\omega}g_{\lambda\mu})B^{\lambda}_{i}B^{\mu}_{j}B^{\omega}_{k}=D_{k}g_{ij}+2\sum_{x}k_{x[j}\overset{x}{\Omega}_{i]k} \tag{3.16}$$

where D_k is the symbolic vector of the covariant derivative with respect to Γ^{k}_{ij}.

Proof. In virtue of (2.14a), (3.1), and (3.3), our assertion follows in the following way:

$$\begin{aligned} D_{k}g_{ij}&=\overset{0}{D}_{k}g_{ij}=\overset{0}{D}_{k}(g_{\lambda\mu}B^{\lambda}_{i}B^{\mu}_{j}) \\ &=(\overset{0}{D}_{k}g_{\lambda\mu})B^{\lambda}_{i}B^{\mu}_{j}+g_{\lambda\mu}((\overset{0}{D}_{k}B^{\lambda}_{i})B^{\mu}_{j}+B^{\lambda}_{i}\overset{0}{D}_{k}B^{\mu}_{j}) \\ &=(D_{\omega}g_{\lambda\mu})B^{\lambda}_{i}B^{\mu}_{j}B^{\omega}_{k}-g_{\lambda\mu}\sum_{x}(\overset{x}{\Omega}_{ik}\underset{x}{N}^{\lambda}B^{\mu}_{j}+\overset{x}{\Omega}_{jk}\underset{x}{N}^{\mu}B^{\lambda}_{i}) \\ &=(D_{\omega}g_{\lambda\mu})B^{\lambda}_{i}B^{\mu}_{j}B^{\omega}_{k}-k_{\lambda\mu}\sum_{x}(-\overset{x}{\Omega}_{ik}B^{\lambda}_{j}\underset{x}{N}^{\mu}+\overset{x}{\Omega}_{jk}B^{\lambda}_{i}\underset{x}{N}^{\mu}) \\ &=(D_{\omega}g_{\lambda\mu})B^{\lambda}_{i}B^{\mu}_{j}B^{\omega}_{k}-\sum_{x}k_{x[j}\overset{x}{\Omega}_{i]k} \end{aligned}$$

The following theorem is an immediate consequence of Theorem 3.9.

Theorem 3.10. On an X_m of X_n, the induced tensor of $\nabla_\omega g_{\lambda\mu} = \nabla_\omega k_{\lambda\mu}$ is given by

$$(\nabla_\omega k_{\lambda\mu}) B_i^\lambda B_j^\mu B_k^\omega = \nabla_k k_{ij} + 2 \sum_x k_{x[j} \overset{x}{\Lambda}_{i]k} \tag{3.17}$$

where ∇_k is the symbolic vector of the covariant derivative with respect to $\left\{ {k \atop ij} \right\}$.

Let ${}^{(2)}\bar{k}_{ij}$ be the induced tensor of ${}^{(2)}k_{\lambda\mu} = k_\lambda{}^\alpha k_{\alpha\mu}$. That is,

$${}^{(2)}\bar{k}_{ij} = {}^{(2)}k_{\lambda\mu} B_i^\lambda B_j^\mu$$

Theorem 3.11. On an X_m of X_n, the induced tensor of ${}^{(2)}k_{\lambda\mu}$ is given by

$$\begin{aligned} {}^{(2)}\bar{k}_{ij} &= {}^{(2)}k_{ij} + \sum_x k_i{}^x k_{xj} \\ &= {}^{(2)}k_{ij} - \sum_x \varepsilon_x k_{ix} k_{jx} \end{aligned} \tag{3.18}$$

where

$${}^{(2)}k_{ij} = k_i^p k_{pj}$$

Proof. In virtue of (2.19), (2.20), (2.24a), and (3.1), the relation (3.18) follows in the following way:

$$\begin{aligned} {}^{(2)}k_{ij} = k_i^p k_{pj} &= k_\lambda{}^\mu k_{\alpha\beta} B_i^\lambda B_j^\beta \left(\delta_\mu^\alpha - \sum_x \underset{x}{N}{}^\alpha \overset{x}{N}_\mu \right) \\ &= {}^{(2)}\bar{k}_{ij} - \sum_x k_i{}^x k_{xj} \end{aligned}$$

In the following two theorems, we prove two necessary and sufficient conditions for Γ_{ij}^k to be Einstein. If the condition (3.19) [or (3.21)] is satisfied on an X_m of X_n, then *the induced connection* Γ_{ij}^k *is an ES connection* in virtue of Theorems 3.6 and 3.12 (or Theorem 3.13).

Theorem 3.12. On an X_m of X_n, the induced connection Γ_{ij}^k is Einstein if and only if the following condition holds:

$$\sum_x (\overset{x}{\Omega}_{ik} k_{jx} - \overset{x}{\Omega}_{kj} k_{ix}) = 0 \tag{3.19}$$

Proof. In virtue of (3.2) and (3.3), we have

$$B_{ij}^\alpha + \Gamma_{\beta\gamma}^\alpha B_i^\beta B_j^\gamma = \Gamma_{ij}^k B_k^\alpha - \sum_x \overset{x}{\Omega}_{ij} \underset{x}{N}{}^\alpha \tag{3.20}$$

Hence, if Γ_{ij}^{k} is Einstein, then the relations (2.32), (2.20), (2.24a), (3.20), (2.1), (2.14a), (2.21), and (2.5a) give

$$
\begin{aligned}
0 &= \partial_k g_{ij} - \Gamma_{ik}^{p} g_{pj} - \Gamma_{kj}^{p} g_{ip} \\
&= \partial_k (g_{\alpha\beta} B_i^{\alpha} B_j^{\beta}) - (B_{ik}^{\varepsilon} + \Gamma_{\alpha\gamma}^{\varepsilon} B_i^{\alpha} B_k^{\gamma}) g_{\lambda\beta} B_j^{\beta} \left(\delta_{\varepsilon}^{\lambda} - \sum_x \underset{x}{N}^{\lambda} \overset{x}{N}_{\varepsilon} \right) \\
&\quad - (B_{kj}^{\varepsilon} + \Gamma_{\gamma\beta}^{\varepsilon} B_k^{\gamma} B_j^{\beta}) g_{\alpha\mu} B_i^{\alpha} \left(\delta_{\varepsilon}^{\mu} - \sum_x \underset{x}{N}^{\mu} \overset{x}{N}_{\varepsilon} \right) \\
&= (\partial_{\gamma} g_{\alpha\beta} - \Gamma_{\alpha\gamma}^{\varepsilon} g_{\varepsilon\beta} - \Gamma_{\gamma\beta}^{\varepsilon} g_{\alpha\varepsilon}) B_i^{\alpha} B_j^{\beta} B_k^{\gamma} \\
&\quad + \left(\Gamma_{ik}^{p} B_p^{\varepsilon} - \sum_y \overset{y}{\Omega}_{ik} \underset{y}{N}^{\varepsilon} \right) k_{\lambda\beta} B_j^{\beta} \sum_x \underset{x}{N}^{\lambda} \overset{x}{N}_{\varepsilon} \\
&\quad + \left(\Gamma_{kj}^{p} B_p^{\varepsilon} - \sum_y \overset{y}{\Omega}_{kj} \underset{y}{N}^{\varepsilon} \right) k_{\alpha\mu} B_i^{\alpha} \sum_x \underset{x}{N}^{\mu} \overset{x}{N}_{\varepsilon} \\
&= \sum_x (\overset{x}{\Omega}_{ik} k_{jx} - \overset{x}{\Omega}_{kj} k_{ix})
\end{aligned}
$$

The converse statement may be proved similarly, taking the reverse steps of the above calculations.

Theorem 3.13. On an X_m of X_n, the induced connection Γ_{ij}^{k} is Einstein if and only if the following condition holds:

$$\sum_x (k_{x[i} \overset{x}{\Omega}_{j]k} - S_{jk}{}^{x} k_{ix}) = 0 \tag{3.21}$$

Proof. If Γ_{ij}^{k} is Einstein, then the relations (2.5b), (3.16), (2.19), (2.20), (2.24a), and (3.1) give

$$
\begin{aligned}
0 &= D_k g_{ij} - 2 S_{kj}{}^{p} g_{ip} \\
&= -2 \sum_x k_{x[j} \overset{x}{\Omega}_{i]k} + (D_{\omega} g_{\lambda\mu} - 2 S_{\omega\mu}{}^{\alpha} g_{\lambda\beta} B_{\alpha}^{\beta}) B_i^{\lambda} B_j^{\mu} B_k^{\omega} \\
&= -2 \sum_x k_{x[j} \overset{x}{\Omega}_{i]k} + 2 \sum_x (S_{\omega\mu}{}^{\alpha} B_k^{\omega} B_j^{\mu} \overset{x}{N}_{\alpha})(g_{\lambda\beta} B_i^{\lambda} \underset{x}{N}^{\beta}) \\
&= 2 \sum_x (k_{x[i} \overset{x}{\Omega}_{j]k} - S_{jk}{}^{x} k_{ix})
\end{aligned}
$$

The following two theorems are direct consequences of the above two theorems.

Theorem 3.14. If the induced connection Γ_{ij}^k of $\Gamma_{\lambda\mu}^\nu$ on an X_m of X_n is Einstein, then the following condition holds:

$$\sum_x (\overset{x}{\Omega}_{[jk]} + S_{jk}{}^x) k_{xi} = 0 \tag{3.22}$$

Proof. If Γ_{ij}^k is Einstein, then the conditions (3.19) and (3.21) hold. Our assertion immediately follows by subtracting (3.19) from (3.21).

Theorem 3.15. Let X_m be a submanifold of X_n with symmetric coefficients $\overset{x}{\Omega}_{ij} = \overset{x}{\Omega}_{ji}$. If the induced connection Γ_{ij}^k is Einstein, then the following condition holds:

$$\sum_x S_{jk}{}^x k_{xi} = 0 \tag{3.23}$$

Proof. This assertion follows immediately from (3.22).

3.3. Analysis on the Submanifolds of ESX_n

In this section, we investigate consequences of the results of the previous section. In virtue of (3.8) and Remark 3.5, we note that on an X_m of ESX_n the following relations always hold:

$$S_{jx}{}^x = 0 \tag{3.24}$$

$$\overset{x}{\Omega}_{ij} = \overset{x}{\Omega}_{ji} \tag{3.25}$$

Theorem 3.16. On an X_m of ESX_n, the induced connection Γ_{ij}^k is of the form

$$\Gamma_{ij}^k = \left\{ {k \atop ij} \right\} + 2\delta_{[i}^k X_{j]} + 2k_{(i}{}^k X_{j)} \tag{3.26}$$

Hence, the induced connection is also semisymmetric.

Proof. This assertion follows by substituting (2.9) into (3.11). Refer to (3.14).

Theorem 3.17. On an X_m of ESX_n, a necessary and sufficient condition for the induced connection Γ_{ij}^k to be Einstein is

$$\sum_x k_{x[i} \overset{x}{\Omega}_{j]k} = 0 \tag{3.27}$$

Proof. This assertion is an immediate consequence of (2.21) and (3.24).

Remark 3.18. In virtue of Theorem 3.17, we note that the condition (3.27) is satisfied on a submanifold X_m of ESX_n which is Einstein. However, it has not been proved that the condition (3.27) is identically satisfied on every X_m of ESX_n. Although this difficulty still exists and it is an open problem, we believe that the following statement is true:

> Every submanifold X_m of ESX_n is Einstein, so that it is also an ES manifold

It seems that this statement may be proved by a line-geometric method.

In our further considerations, we use the symbol ESX_m to denote those submanifolds X_m of ESX_n which satisfy the condition (3.27).

Remark 3.19. Note that the condition (3.19) or (3.27) was already obtained by Chung *et al.* (1989, p. 864) and named the *ES identity* in Theorem 4.5, stating that on every X_m of ESX_n the condition (3.27) holds identically. However, as we see in Remark 3.18, it is a wrong result at the present time. This wrong result had been obtained from the misassumption that on an X_m of ESX_n the induced tensor of $D_\omega g_{\lambda\mu}$ is $D_k g_{ij}$. But this was found to be a wrong assumption in virtue of (3.16). As we remarked in Remark 3.18, there is no way to prove that the second term of the right-hand side of (3.16) identically vanishes on every X_m of ESX_n.

Theorem 3.20. On an ESX_m of ESX_n, the following relations hold:

$$\sum_x k_{x[i}\overset{x}{\Omega}_{j]k}=0 \tag{3.28}$$

$$\sum_x k_{x[i}\overset{x}{\Lambda}_{j]k}=\sum_x k_k{}^x k_{x[i}X_{j]} \tag{3.29}$$

$$D_k g_{ij}=(D_\omega g_{\lambda\mu})B_i^\lambda B_j^\mu B_k^\omega \tag{3.30}$$

$$\nabla_k k_{ij}=(\nabla_\omega k_{\lambda\mu})B_i^\lambda B_j^\mu B_k^\omega+2\sum_x k_k{}^x k_{x[j}X_{i]} \tag{3.31}$$

Proof. Since ESX_m is Einstein, the relations (3.28) and (3.30) are respectively direct consequences of (3.27) and (3.16). The relation (3.31) follows from (3.17) and (3.29). In order to prove the relation (3.29), we first note that

$$-k_{xi}k_j{}^x+k_{xj}k_i{}^x=\varepsilon_x(-k_{xi}k_{jx}+k_{xj}k_{ix})=0$$

The relation (3.29) immediately follows by substituting (3.7b) into (3.28) and using the above relation.

Theorem 3.21. On an ESX_m of ESX_n, the following identity, the corresponding induced equation of (2.11), holds:

$$\nabla_k k_{ij}+2P_{k[i}X_{j]}=0 \tag{3.32}$$

where

$$P_{ij} = {}^{(2)}k_{ij} - h_{ij} \tag{3.33}$$

Proof. Let $\bar{P}_{ij}$ be the induced tensor of $P_{\omega\lambda}$. Multiplying by $B_i^\lambda B_k^\omega$ on both sides of (2.12a) and making use of (3.18) and (3.23), we have

$$\bar{P}_{ki} = P_{\omega\lambda} B_i^\lambda B_k^\omega = {}^{(2)}\bar{k}_{ki} - h_{ki} = P_{ki} - \sum_x k_k{}^x k_{xi} \tag{3.34}$$

Hence, multiplying by $B_i^\lambda B_j^\mu B_k^\omega$ on both sides of (2.11) and making use of (3.31) and (3.34), we have that equation (3.32) follows:

$$\begin{aligned} 0 &= (\nabla_\omega k_{\lambda\mu}) B_i^\lambda B_j^\mu B_k^\omega + 2\bar{P}_{k[i} X_{j]} \\ &= (\nabla_\omega k_{\lambda\mu}) B_i^\lambda B_j^\mu B_k^\omega + 2P_{k[i} X_{j]} - 2\sum_x k_k{}^x k_{x[i} X_{j]} \\ &= \nabla_k k_{ij} + 2P_{k[i} X_{j]} \end{aligned}$$

4. THE GENERALIZED FUNDAMENTAL EQUATIONS FOR SUBMANIFOLDS OF ESX_n

This section is devoted to the derivation of the generalized fundamental equations for submanifolds of ESX_n, such as the generalized Weingarten equations and Gauss–Codazzi equations. The generalized Gauss formulas were already obtained in (3.10). Formally, we state the following result.

Theorem 4.1. (The generalized Gauss formulas for an X_m of ESX_n.) On an X_m of ESX_n, the following relations hold:

$$\overset{0}{D}_j B_i^\alpha = \sum_x (-\overset{x}{\Lambda}_{ij} + 2\varepsilon_x X_{(i} k_{j)x}) \underset{x}{N}^\alpha \tag{4.1}$$

In order to derive the generalized Weingarten equations, we need the following preparations.

Let

$$\underset{jx}{\mathsf{M}}{}^\alpha = \overset{0}{D}_j \underset{x}{N}^\alpha \tag{4.2}$$

Theorem 4.2. The vector $\underset{jx}{\mathsf{M}}{}^\alpha$ may be decomposed as

$$\underset{jx}{\mathsf{M}}{}^\alpha = \underset{jx}{\mathsf{M}}{}^i B_i^\alpha + \sum_y \underset{jx}{\mathsf{M}}{}^y \underset{y}{N}^\alpha \tag{4.3}$$

the first vector being tangential to X_m and the second normal to X_m. Furthermore, $\underset{jx}{\mathsf{M}}{}^i$ is also the induced tensor of $D_\gamma \underset{x}{N}^\alpha$ and $\underset{jx}{\mathsf{M}}{}^y$ is the induced vector of $(D_\gamma \underset{x}{N}^\alpha)\overset{y}{N}_\alpha$. That is,

$$\underset{jx}{\mathsf{M}}^i = \underset{jx}{\mathsf{M}}^\alpha B^i_\alpha = (D_\gamma \underset{x}{N}^\alpha) B^i_\alpha B^\gamma_j \tag{4.4a}$$

$$\underset{jx}{\mathsf{M}}^y = \underset{jx}{\mathsf{M}}^\alpha \overset{y}{N}_\alpha = ((D_\gamma \underset{x}{N}^\alpha)\overset{y}{N}_\alpha) B^\gamma_j \tag{4.4b}$$

Proof. The first assertion (4.3) follows from Theorem 2.4. The relations (4.4) are obvious in virtue of (2.19).

Theorem 4.3a. On an X_m of ESX_n, the induced vector $\underset{jx}{\mathsf{M}}^i$ of $\underset{jx}{\mathsf{M}}^\alpha$ is given by

$$\underset{jx}{\mathsf{M}}^i = \varepsilon_x h^{im} \overset{x}{\Lambda}_{mj} + 2k_{(x}{}^i X_{j)} + \delta^i_j X_x \tag{4.5a}$$

Proof. In virtue of (4.4a), (2.10), and (2.21), we have

$$\begin{aligned}\underset{jx}{\mathsf{M}}^i &= [\partial_\gamma \underset{x}{N}^\beta + (\{{}^{\,\beta}_{\varepsilon\gamma}\} + 2\delta^\beta_{[\varepsilon} X_{\gamma]} + 2k_{(\varepsilon}{}^\beta X_{\gamma)}) \underset{x}{N}^\varepsilon] B^i_\beta B^\gamma_j \\ &= (\nabla_\gamma \underset{x}{N}^\beta) B^i_\beta B^\gamma_j - X_x \delta^i_j + 2k_{(\varepsilon}{}^\beta X_{\gamma)} \underset{x}{N}^\varepsilon B^i_\beta B^\gamma_j \end{aligned} \tag{4.6}$$

Using (2.22a), (2.23), and (3.6), the first term of (4.6) may be written as

$$\begin{aligned}(\text{first term}) &= (\nabla_\gamma \underset{x}{N}_\varepsilon) h^{\beta\varepsilon} B^i_\beta B^\gamma_j \\ &= \varepsilon_x h^{im} (\nabla_\gamma \overset{x}{N}_\varepsilon) B^\varepsilon_m B^\gamma_j = \varepsilon_x h^{im} \overset{x}{\Lambda}_{mj}\end{aligned} \tag{4.7}$$

In virtue of (2.23), the third term of (4.6) is

$$\begin{aligned}(\text{third term}) &= (k_\varepsilon{}^\beta \underset{x}{N}^\varepsilon B^i_\beta)(X_\gamma B^\gamma_j) + (k_\gamma{}^\beta B^i_\beta B^\gamma_j)(X_\varepsilon \underset{x}{N}^\varepsilon) \\ &= h^{im} k_{xm} X_j + k_j{}^i X_x\end{aligned} \tag{4.8}$$

We now substitute (4.7) and (4.8) into (4.6) to obtain (4.5a).

Theorem 4.3b. On an X_m of ESX_n, the induced vector $\underset{jx}{\mathsf{M}}^i$ of $\underset{jx}{\mathsf{M}}^\alpha$ is given by

$$\underset{jx}{\mathsf{M}}^i = \varepsilon_x h^{im} \overset{x}{\Omega}_{mj} + k_{jx} X^i - \delta^i_j X_x + k_j{}^i X_x \tag{4.5b}$$

Proof. In virtue of (3.7b), the first term of (4.5a) may be written as

$$\varepsilon_x h^{im} \overset{x}{\Lambda}_{mj} = \varepsilon_x h^{im} \overset{x}{\Omega}_{mj} + 2h^{im} X_{(m} k_{j)x}$$

Now, the representation (4.5b) follows by substituting the above relation into (4.5a) and making use of the skew-symmetry of k_{mx}.

The following abbreviation will be used in our further considerations:

$$\overset{y}{\underset{x}{\mathsf{H}}}{}_{\gamma}=\varepsilon_y(\nabla_{\gamma}\underset{x}{N}{}^{\alpha})\overset{y}{N}_{\alpha} \tag{4.9}$$

Theorem 4.4. The tensor $\overset{y}{\underset{x}{\mathsf{H}}}{}_{\gamma}$ satisfies the relation

$$\overset{y}{\underset{x}{\mathsf{H}}}{}_{\gamma}+\overset{x}{\underset{y}{\mathsf{H}}}{}_{\gamma}=0 \tag{4.10a}$$

In particular,

$$\overset{x}{\underset{x}{\mathsf{H}}}{}_{\gamma}=0 \tag{4.10b}$$

Proof. The relation (4.10b) is a direct consequence of (4.10a). The relation (4.10a) follows from (4.9) and

$$0=\nabla_{\gamma}(h_{\alpha\beta}\underset{x}{N}{}^{\alpha}\underset{y}{N}{}^{\beta})=\varepsilon_y(\nabla_{\gamma}\underset{x}{N}{}^{\alpha})\overset{y}{N}_{\alpha}+\varepsilon_x(\nabla_{\gamma}\underset{y}{N}{}^{\alpha})\overset{x}{N}_{\alpha}$$

Theorem 4.5. On an X_m of ESX_n, the C-nonholonomic components $\underset{jx}{\mathsf{M}}{}^{y}$ of $\underset{jx}{\mathsf{M}}{}^{\alpha}$ are given by

$$\underset{jx}{\mathsf{M}}{}^{y}=\varepsilon_y\overset{y}{\underset{x}{\mathsf{H}}}{}_{\gamma}B_j^{\gamma}+\delta_x^y X_j+2k_{(j}{}^{y}X_{x)} \tag{4.11}$$

Proof. Making use of (2.10), (4.9), (2.21), and (2.26), we can obtain the expression (4.11) obtained from (4.4b) as follows:

$$\begin{aligned}
\underset{jx}{\mathsf{M}}{}^{y}&=(D_{\gamma}\underset{x}{N}{}^{\beta})\overset{y}{N}_{\beta}B_j^{\gamma}\\
&=(\nabla_{\gamma}\underset{x}{N}{}^{\beta})\overset{y}{N}_{\beta}B_j^{\gamma}+2(\delta_{[\alpha}{}^{\beta}X_{\gamma]}+k_{(\alpha}{}^{\beta}X_{\gamma)})\underset{x}{N}{}^{\alpha}\overset{y}{N}_{\beta}B_j^{\gamma}\\
&=\varepsilon_y\overset{y}{\underset{x}{\mathsf{H}}}{}_{\gamma}B_j^{\gamma}+\delta_x^y(X_{\gamma}B_j^{\gamma})+(k_{\alpha}{}^{\beta}\underset{x}{N}{}^{\alpha}\overset{y}{N}_{\beta})(X_{\gamma}B_j^{\gamma})+(k_{\gamma}{}^{\beta}B_j^{\gamma}\overset{y}{N}_{\beta})(X_{\alpha}\underset{x}{N}{}^{\alpha})\\
&=\varepsilon_y\overset{y}{\underset{x}{\mathsf{H}}}{}_{\gamma}B_j^{\gamma}+\delta_x^y X_j+k_x{}^{y}X_j+k_j{}^{y}X_x
\end{aligned}$$

Now, we are ready to present the following two representations of the generalized Weingarten equations for an X_m of ESX_n, by simply substituting (4.5a), (4.5b), and (4.11) into (4.3).

Theorem 4.6. On a X_m of ESX_n, we have

$$\overset{0}{D}_j \underset{x}{N}^\alpha = (\varepsilon_x h^{im} \overset{x}{\Lambda}_{mj} + 2X_{(x}k_{j)}{}^i - \delta_j^i X_x) B_i^\alpha + \sum_y (\varepsilon_y \underset{x}{\overset{y}{\mathrm{H}}}{}_\gamma B_j^\gamma + \delta_x^y X_j + 2k_{(j}{}^y X_{x)}) \underset{y}{N}^\alpha \tag{4.12a}$$

(The first representation of the generalized Weingarten equations on an X_m of ESX_n)

$$\overset{0}{D}_j \underset{x}{N}^\alpha = (\varepsilon_x h^{im} \overset{x}{\Omega}_{mj} + k_{jx} X^i - \delta_j^i X_x + k_j{}^i X_x) B_i^\alpha + \sum_y (\varepsilon_y \underset{x}{\overset{y}{\mathrm{H}}}{}_\gamma B_j^\gamma + \delta_x^y X_j + 2k_{(j}{}^y X_{x)}) \underset{y}{N}^\alpha \tag{4.12b}$$

(The second representation of the generalized Weingarten equations on an X_m of ESX_n)

In the derivation of the generalized Gauss–Codazzi equations, we need the following curvature tensors $R_{\omega\mu\lambda}{}^\nu$ of ESX_n and $R_{ijk}{}^h$ of X_m:

$$R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu}\Gamma^\nu_{|\lambda|\omega]} + \Gamma^\alpha_{\lambda[\omega}\Gamma^\nu_{|\alpha|\mu]}) \tag{4.13}$$

$$R_{ijk}{}^h = 2(\partial_{[j}\Gamma^h_{|k|i]} + \Gamma^p_{k[i}\Gamma^h_{|p|j]}) \tag{4.14}$$

Theorem 4.7. (The generalized Gauss–Codazzi equations for an X_m of ESX_n.) On an X_m of ESX_n, the curvature tensors defined by (4.13) and (4.14) are involved in the following identities:

$$R_{ijk}{}^h = R_{\beta\gamma\varepsilon}{}^\alpha B_i^\beta B_j^\gamma B_k^\varepsilon B_\alpha^h + 2\sum_x \overset{x}{\Omega}_{k[i}(\overset{x}{\Omega}_{j]m}\varepsilon_x h^{hm} - \delta_{j]}^h X_x + k_{j]}{}^h X_x + k_{j]x} X^h) \tag{4.15}$$

(The generalized Gauss equations for an X_m of ESX_n)

$$2\overset{0}{D}_{[k}\overset{x}{\Omega}_{j]i} = R_{\beta\gamma\varepsilon}{}^\alpha B_i^\varepsilon B_j^\gamma B_k^\beta \underset{}{\overset{x}{N}}_\alpha + 6\overset{x}{\Omega}_{i[k}X_{j]} + 2\sum_y \overset{y}{\Omega}_{i[k}(B_{j]}^\gamma \varepsilon_x \underset{y}{\overset{x}{\mathrm{H}}}{}_\gamma + X_{j]}k_y{}^x + k_{j]}{}^x X_y) \tag{4.16}$$

(The generalized Codazzi equations for an X_m of ESX_n)

Proof. In virtue of (3.2), (3.3), (4.13), and (4.14), we have

$$2\overset{0}{D}_{[k}\overset{0}{D}_{j]}B_i^\alpha = 2[\partial_{[k}(\overset{0}{D}_{j]}B_i^\alpha) - \Gamma_{[jk]}(\overset{0}{D}_m B_i^\alpha)$$

$$-\Gamma^m_{i[k}(\overset{0}{D}_{j]}B_m^\alpha) + \Gamma^\alpha_{B\gamma}(\overset{0}{D}_{[j}B^\beta_{|i|})B^\gamma_{k]}]$$

$$= -R_{\varepsilon\gamma\beta}{}^\alpha B_i^\beta B_j^\gamma B_k^\varepsilon + R_{kji}{}^m B_m^\alpha + 4\sum_x \overset{x}{\Omega}_{i[j}X_{k]}\underset{x}{N}{}^\alpha \tag{4.17}$$

where use of the relation $S_{jk}{}^m = 2\delta_{[j}{}^m X_{k]}$ has been made in the above lengthy calculations. On the other hand, the relations (3.3) and (4.12a) and the symmetry of $\overset{x}{\Omega}_{ij}$ give

$$2\overset{0}{D}_{[k}\overset{0}{D}_{j]}B_i^\alpha = -2\sum_x \overset{0}{D}_{[k}(\overset{x}{\Omega}_{j]i}\underset{x}{N}{}^\alpha)$$

$$= 2\sum_x (\overset{0}{D}_{[j}\overset{x}{\Omega}_{k]i})\underset{x}{N}{}^\alpha + 2\sum_x \overset{x}{\Omega}_{i[k}\overset{0}{D}_{j]}\underset{x}{N}{}^\alpha$$

$$= 2\sum_x (\overset{0}{D}_{[j}\overset{x}{\Omega}_{k]i} + \overset{x}{\Omega}_{i[k}X_{j]})\underset{x}{N}{}^\alpha$$

$$+ 2\sum_{x,y} \overset{x}{\Omega}_{i[k}(B^\gamma_{j]}\varepsilon_y \underset{x}{\overset{y}{H}}{}_\gamma + X_{j]}k_x{}^y + k_{j]}{}^y X_x)\underset{y}{N}{}^\alpha$$

$$+ 2\sum_x \overset{x}{\Omega}_{i[k}(\overset{x}{\Omega}_{j]m}\varepsilon_x h^{pm} - \delta^p_{j]}X_x + k_{j]x}X^p + k_{j]}{}^p X_x)B_p^\alpha \tag{4.18}$$

Hence, comparing (4.17) and (4.18), we have

$$R_{kji}{}^m B_m^\alpha = R_{\beta\gamma\varepsilon}{}^\alpha B_k^\beta B_j^\gamma B_i^\varepsilon + 2\sum_x (\overset{0}{D}_{[j}\overset{x}{\Omega}_{k]i} + 3\overset{x}{\Omega}_{i[k}X_{j]})\underset{x}{N}{}^\alpha$$

$$+ 2\sum_{x,y} \overset{x}{\Omega}_{i[k}(B^\gamma_{j]}\varepsilon_x \underset{x}{\overset{y}{H}}{}_\gamma + X_{j]}k_x{}^y + k_{j]}{}^y X_x)\underset{y}{N}{}^\alpha$$

$$+ 2\sum_x \overset{x}{\Omega}_{i[k}(\overset{x}{\Omega}_{j]m}\varepsilon_x h^{pm} - \delta^p_{j]}X_x + k_{j]x}X_x + k_{j]}{}^p X_x)B_p^\alpha \tag{4.19}$$

Making use of (2.21), the identity (4.15) follows by multiplying both sides of (4.19) by B^h_α and interchanging the indices i and k. Similarly, multiplying $\underset{z}{N}_\alpha$ into both sides of (4.19) and replacing the indices x by y and z by x, we have (4.16).

Remark 4.8. Note, in particular, that *on an ESX_m of ESX_n* the terms

$$2\sum_{x}\overset{x}{\Omega}_{k[i}k_{j]x}X^h \text{ of (4.15)} \qquad \text{and} \qquad 2\sum_{y}\overset{x}{\Omega}_{i[k}k_{j]}{}^{x}X^h \text{ of (4.16)}$$

vanish in virtue of the identity (3.28).

5. TWO SPECIAL SUBMANIFOLDS OF ESX_n

In this section, we introduce two special submanifolds of ESX_n, namely, hypersubmanifolds and T submanifolds, and investigate their properties with particular emphasis on the specialization of the results obtained in the previous section.

5.1. Hypersubmanifolds of ESX_n

When the dimension of X_m is $m=n-1$ (namely, for the case of *hypersubmanifolds*), the theory of submanifolds assumes a particularly simple and geometrically illuminating form. This simplification is mainly due to the fact that under this circumstances there exists a unique normal N^α at each point of X_{n-1}.

In this case, quantities used in the previous sections take the following simpler forms and values:

$$\varepsilon_x=1 \tag{5.1a}$$

$$\underset{x}{N}^\alpha=\underset{n}{N}^\alpha \overset{\text{def}}{=} N^\alpha, \qquad \overset{y}{N}_\alpha=\overset{n}{N}_\alpha \overset{\text{def}}{=} N_\alpha \tag{5.1b}$$

$$\overset{x}{\Omega}_{ij}=\overset{n}{\Omega}_{ij} \overset{\text{def}}{=} \Omega_{ij}, \qquad \overset{x}{\Lambda}_{ij}=\overset{n}{\Lambda}_{ij} \overset{\text{def}}{=} \Lambda_{ij} \tag{5.1c}$$

$$X_x=X^x=X_\alpha N^\alpha \overset{\text{def}}{=} \phi \tag{5.1d}$$

$$k_{ix}=k_i{}^x=k_{in} \overset{\text{def}}{=} k_i \tag{5.1e}$$

$$k_{xy}=k_x{}^y=k_{nn}=0 \tag{5.1f}$$

$$\underset{x}{\overset{y}{\mathsf{H}}}{}_\gamma=\underset{n}{\overset{n}{\mathsf{H}}}{}_\gamma=0 \tag{5.1g}$$

It may be easily checked that

$$k_x{}^i=-k^i \tag{5.2}$$

Theorem 5.1. (The generalized fundamental equations for an X_{n-1} of

ESX_n.) On an X_{n-1} of ESX_n, the following identities hold:

$$\overset{0}{D}_j B_i^\alpha = -\Omega_{ij} N^\alpha = (-\Lambda_{ij} + 2X_{(i}k_{j)}) N^\alpha \tag{5.3}$$

(Generalized Gauss formulas)

$$\begin{aligned}\overset{0}{D}_j N^\alpha &= (h^{im}\Lambda_{mj} - X_j k^i - \phi\delta_j^i + \phi k_j{}^i) B_i^\alpha + (\phi k_j + X_j) N^\alpha \\ &= (h^{im}\Omega_{mj} + X^i k_j - \phi\delta_j^i + \phi k_j{}^i) B_i^\alpha + (\phi k_j + X_j) N^\alpha \end{aligned}\tag{5.4}$$

(Generalized Weingarten equations)

$$R_{ijk}{}^h = R_{\beta\gamma\varepsilon}{}^\alpha B_i^\beta B_j^\gamma B_k^\varepsilon B_\alpha^h + 2\Omega_{k[i}(\Omega_{j]m}h^{hm} - \phi\delta_{j]}^h + \phi k_{j]}{}^h + k_{j]}X^h) \tag{5.5}$$

(Generalized Gauss equations)

$$2\overset{0}{D}_{[k}\Omega_{j]i} = R_{\beta\gamma\varepsilon}{}^\alpha B_i^\beta B_j^\gamma B_k^\varepsilon N_\alpha + 6\Omega_{i[k}X_{j]} + 2\phi\Omega_{i[k}k_{j]} \tag{5.6}$$

(Generalized Codazzi equations)

Proof. In virtue of (5.1) and (5.2), the relations in this theorem follow from (4.1), (4.12), (4.15), and (4.16), respectively.

Remark 5.2. Note, in particular, that the Gauss–Codazzi equations *on an ES submanifold of* ESX_n are the relations (5.5) and (5.6) with the vanishing last terms in each equation, since in this case the identity (3.28) is reduced to

$$k_{[i}\Omega_{j]k} = 0 \tag{5.7}$$

Remark 5.3. In virtue of (2.18b), (2.15), (2.17), and (2.21), it may be shown that the following relations hold on an X_m of ESX_n:

$$k_\beta{}^\alpha \underset{x}{N}^\beta = k_x{}^i B_i^\alpha + \sum_y k_x{}^y \underset{y}{N}^\alpha \tag{5.8a}$$

$$k_\beta{}^\alpha B_j^\beta = k_j{}^i B_i^\alpha + \sum_y k_j{}^y \underset{y}{N}^\alpha \tag{5.8b}$$

Using the relations (5.8) together with (5.1) and (5.2), it may be easily checked that the generalized fundamental equations presented in Theorem 5.1 coincide with those obtained in Chung and Lee (1989).

5.2. *T*-Submanifolds of ESX_n

A submanifold X_m of ESX_n whose *ES* vector X_λ is tangential to X_m at each point of X_m will be called a *tangential submanifold* of ESX_n and will be denoted by TX_m. The simplification of TX_m is due to the fact that it satisfies

$$X_\lambda = X_i B^i_\lambda, \qquad X_x = 0 \tag{5.9a}$$

$$X^\nu = X^i B^\nu_i, \qquad X^x = 0 \tag{5.9b}$$

In fact, *an X_m of ESX_n is a TX_m if and only if any one of the relations in* (5.9) *holds.*

The following theorem gives an alternative characterization of TX_m.

Theorem 5.4. A necessary and sufficient condition for the *ES* vector X_λ to be tangential to X_m of ESX_n at each point of X_m is that the basic tensor $g_{\lambda\mu}$ satisfies the following condition:

$$(\nabla_\omega k_{\alpha\beta}) \underset{x}{N}^\alpha \underset{y}{N}^\beta = 0 \qquad \text{for all } x, y \tag{5.10a}$$

or equivalently

$$(\nabla_\omega k^{\alpha\beta}) \overset{x}{N}_\alpha \overset{y}{N}_\beta = 0 \qquad \text{for all } x, y \tag{5.10b}$$

Proof. Suppose that the vector X_λ is tangential to X_m. The condition (5.10a) immediately follows by multiplying by $\underset{x}{N}^\lambda \underset{y}{N}^\mu$ on both sides of (2.11).

Conversely, suppose that the condition (5.10a) holds for $g_{\lambda\mu}$. In order to prove that the vector X_λ is tangential to X_m, it suffices to show that $X_x = 0$. Let

$$T_{x\omega} = \underset{x}{N}^\alpha P_{\omega\alpha}$$

We first note that the $(n-m)\times n$ matrix $(\underset{x}{N}^\alpha)$ and $n \times n$ matrix $(P_{\omega\alpha})$ are respectively of rank $n-m$ and n. Now, multiply by $\underset{x}{N}^\lambda \underset{y}{N}^\mu$ on both sides of (2.11) to obtain

$$T_{x\omega} X_y = T_{y\omega} X_x \qquad \text{for all } x, y$$

which is an identity for $x=y$. If we assume that $X_x \neq 0$ for all x, we must have

$$T_{y\omega} = \frac{X_y}{X_x} T_{x\omega} \qquad \text{for all } x \neq y$$

This implies that the rank of the matrix $(T_{x\omega})$ is less than $n-m$, which is a contradiction to our previous discussion. Therefore, we have

$$X_x = X_\alpha \underset{x}{N}^\alpha = 0 \qquad \text{for all } x$$

The equivalence of (5.10a) and (5.10b) is obvious.

Now, in virtue of (5.1) and (5.9), the following theorem follows from (4.1), (4.12), (4.15), and (4.16).

Theorem 5.5. (The generalized fundamental equations for tangential submanifolds of ESX_n.) On each of the following tangential submanifolds of ESX_n, we have

$$\overset{0}{D}_j B_i^\alpha = \begin{cases} \sum_x (-\overset{x}{\Lambda}_{ij} + 2\varepsilon_x X_{(i} k_{j)x}) \underset{x}{N}^\alpha & \text{on all } TX_m \\ (-\Lambda_{ij} + 2X_{(i} k_{j)}) N^\alpha & \text{on all } TX_{n-1} \end{cases} \tag{5.11}$$

(The generalized Gauss formulas)

$$\overset{0}{D}_j \underset{x}{N}^\alpha = \begin{cases} \sum_y (\varepsilon_y \underset{x}{\overset{y}{\mathrm{H}}}{}_\gamma B_j^\gamma + X_j k_x{}^y + X_j \delta_x^y) \underset{y}{N}^\alpha \\ \quad + (\varepsilon_x h^{im} \overset{x}{\Lambda}_{mj} + X_j k_x{}^i) B_i^\alpha & \text{on all } TX_m \\ (h^{im} \Lambda_{mj} - X_j k^i) B_i^\alpha + X_j N^\alpha & \text{on all } TX_{n-1} \end{cases} \tag{5.12a}$$

(The first representation of generalized Weingarten equations)

$$\overset{0}{D}_j \underset{x}{N}^\alpha = \begin{cases} \sum_y (\varepsilon_y \underset{x}{\overset{y}{\mathrm{H}}}{}_\gamma B_j^\gamma + X_j k_x{}^y + X_j \delta_x^y) \underset{y}{N}^\alpha \\ \quad + (\varepsilon_x h^{im} \overset{x}{\Omega}_{mj} + X^i k_{jx}) B_i^\alpha & \text{on all } TX_m \\ (h^{im} \Omega_{mj} + X^i k_j) B_i^\alpha + X_j N^\alpha & \text{on all } TX_{n-1} \end{cases} \tag{5.12b}$$

(The second representation of generalized Weingarten equations)

$$R_{ijk}{}^h = R_{\beta\gamma\varepsilon}{}^\alpha B_i^\beta B_j^\gamma B_k^\varepsilon B_\alpha^h + \begin{cases} 2\sum_x \overset{x}{\Omega}_{k[i} (\overset{x}{\Omega}_{j]m} \varepsilon_x h^{hm} + k_{j]x} X^h) & (\text{on } TX_m) \\ f2\sum_x \varepsilon_x h^{hm} \overset{x}{\Omega}_{k[i} \overset{x}{\Omega}_{j]m} & (\text{on } TESX_m) \\ 2\Omega_{k[i} (\Omega_{j]m} h^{hm} + k_{j]} X^h) & (\text{on } TX_{n-1}) \\ 2h^{hm} \Omega_{k[i} \Omega_{j]m} & (\text{on } TESX_{n-1}) \end{cases} \tag{5.13}$$

(The generalized Gauss equations)

$$2\overset{0}{D}_{[k} \overset{x}{\Omega}_{j]i} = R_{\beta\gamma\varepsilon}{}^\alpha B_k^\beta B_j^\gamma B_i^\varepsilon \overset{x}{N}_\alpha + \begin{cases} 6\overset{x}{\Omega}_{i[k} X_{j]} + 2\sum_y \overset{x}{\Omega}_{i[k} (B_{j]}^\gamma \varepsilon_x \underset{x}{\overset{y}{\mathrm{H}}}{}_\gamma + X_{j]} k_y{}^x) & (\text{on all } TX_m) \\ 6\Omega_{i[k} X_{j]} & (\text{on all } TX_{n-1}) \end{cases} \tag{5.14}$$

(The generalized Codazzi equations)

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